Systemic Contingent Claims Analysis

Abstract

This methodological framework was developed to estimate potential extreme losses in the banking system based on expected losses of systemically important banks and interbank linkages. It describes a systemic risk analysis model designed to assess the magnitude of systemic risk. The model consists of two main stages: conducting conditional risk analysis for systemically important banks and aggregating banks' expected losses into a multidimensional distribution for the banking system as a whole.

Keywords: bank capital, default, expected losses, risk margin, volatility, extreme value, probability.

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Systemic Contingent Claims Analysis

The Systemic Contingent Claims Analysis[1] model is used to assess systemic risks by analyzing the relationship between the market value of bank assets and liabilities. In this model, using expected losses of assets in banks and interconnectedness between banks, expected shortfall for the banking system is determined.

The Systemic Contingent Claims Analysis model is implemented in two steps. In the first step, the Contingent Claims Analysis[2] is conducted using the factors representing bank distress. In this context, DtD, PD and expected loss are determined for each analyzed bank.

In the second step, the estimated expected losses for banks are combined into a multivariate distribution at the banking system level. The probabilistic measure of banks' joint expected losses is determined, and the Systemic Contingent Claims Analysis value for the banking system is derived.

A. Contingent Claims Analysis

Through Contingent Claims Analysis, the probability of a decline in the bank assets below distress barrier and the expected losses at that probability are estimated based on balance sheet and market data. Specifically, in implementing Contingent Claims Analysis, the distress barrier is identified based on the market value of bank equity, equity volatility, the risk-free rate, and outstanding liabilities. These indicators are then used to calculate the DtD, PD, and expected losses.

The market value of bank equity is determined as the sum of the products of the market prices and the number of outstanding ordinary and preference shares on the capital market. The volatility of bank's equity market value is measured as the annualized SD[3] of daily volatility over a 125-day rolling window. The rate on treasury bonds is used as the risk-free rate.

In the Contingent Claims Analysis model, the distress barrier is defined. A decline in bank assets below this barrier implies an increase in the PD. In this context, the

[1] Jobst, A., & Gray, D. (2013). Systemic Contingent Claims Analysis – Estimating Market-Implied Systemic Risk. IMF Working Paper.

[2] Kyriakopoulos, C., Koulis, A., & Varvounis, G. (2023). Importance of the Contingent Claims Analysis in Detecting Banking Risks: Evidence from the Greek Bank Crisis. Journal of Central Banking Theory and Practice.

[3] Taking into account the average of 250 trading days per year, the volatility of the bank equity market value is annualized by multiplying the SD of the daily volatility calculated over the 125-day rolling window by $\sqrt{250}$.

distress barrier is specified as the sum of short-term debts and one-half of long-term debts:

$$B = D_S + 0{,}5 * D_L$$

Where,
B – distress barrier;
D_S – short-term debts;
D_L – long-term debts.

The value of bank's risky debt is determined by the difference between the current value of default-free debt and the expected losses:

$$D_t = B_t e^{-(T-t)} - P_t$$

Where,
D_t – value of risky debt at time t;
B_t – distress barrier;
T – maturity;
P_t – value of expected losses at time t.

Figure 1. Contingent Claims Analysis Recursions

Input Data

$$E, \sigma_E, B, r, T$$

↓

Assessment of the Market Value and Volatility of Assets

$$A, \sigma_A$$

↓

Determination of DtD

$$d_1 = \frac{ln\left(\frac{A}{B}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}}$$

$$d_2 = d_1 - \sigma_A\sqrt{T}$$

↓

Estimation of PD

$$PD = N(-d_2) = 1 - N(d_2)$$

↓

Estimation of Expected Losses

$$P = B_T e^{-rT}\big(1 - N(d_2)\big) - A\big(1 - N(d_1)\big)$$

Source: CBU.

The assets usually follow a stochastic process[4] and their differential equation can be represented as follows:

$$\frac{dA}{A} = \mu_A dt + \sigma_A dZ(t)$$

Where,
$\frac{dA}{A}$ – relative change in asset value;
μ_A – instantaneous drift;
σ_A – volatility of assets;
$Z(t)$ – Wiener-process[5].

The differential equation of a continuous random variable[6] presented above is referred as a geometric Brownian motion[7].

By solving the geometric Brownian motion equation, the continuous random variable of assets at period t is expressed as follows:

$$A_t = A_0 e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)t + \sigma_A Z(t)}$$

Where,
A_t – market value of assets at time t;
A_0 – initial value of assets;
μ_A – instantaneous drift;
σ_A – volatility of assets;
$Z(t)$ – Wiener-process.

[4] Stochastic process is a process that evolves over time under the influence of randomness.

[5] Szabados, T. (1994). An Elementary Introduction to the Wiener Process and Stochastic Integrals. Technical University of Budapest.

Wiener-process represents the stochastic component of asset prices and is used to model unpredictable random fluctuations in asset prices.

[6] The continuous random variable is a random variable that can take all values within a finite or infinite interval, and its probability density function is expressed in the following form:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

Where,
a – lower bound of the probability density function;
b – upper bound of the probability density function.

[7] Shreve, S. (2004). Stochastic Calculus for Finance II: Continuous-Time Models. Springer Finance.

Geometric Brownian motion is a model that describes the evolution of financial asset prices over time.

The value of the bank's assets at time t falling below the distress barrier indicates vulnerabilities. The theoretically derived probability of this event is expressed as follows:

$$P(A_t \leq B_t) = N\left(-\frac{\ln\left(\frac{A_0}{B_t}\right) + \left(\mu_A - \frac{\sigma_A^2}{2}\right)(T-t)}{\sigma_A\sqrt{T-t}} = -d_{2,\mu}\right).$$

Where,
$P(A_t \leq B_t)$ – probability of distress;;
$N(-d_{2,\mu})$ – standard cumulative normal distribution function;
$d_{2,\mu}$ – DtD.

Using the sum of capital and debt equal to the value of assets, the value of risky debt is determined as follows:

$$D_t = A_t - E_t.$$

Where,
D_t – market value of risky debt;
E_t – market value of equity;
A_t – market value of assets.

In order to determine the market value of debt, the market value of the assets and their volatility are used:

$$E = AN(d_1) - Be^{-rT}N(d_2).$$
$$D = A - AN(d_1) + Be^{-rT}N(d_2)$$
$$d_1 = \frac{ln\left(\frac{A}{B}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}}$$
$$d_2 = d_1 - \sigma_A\sqrt{T}$$

The market value and volatility of assets are determined by the market value and volatility of equity applying Ito's lemma[8]. Ito's lemma helps to calculate the

[8] Washburn, B., & Dik, M. (2021). Derivation of Black-Scholes equation using Ito's lemma. Proceedings of International Mathematical Sciences.

variation of a function in continuously evolving and unpredictable processes. In this context, the following equation is obtained by adding a second-order term to the ordinary differential:

$$dE = \left(\frac{\partial E}{\partial A}\mu_A + \frac{\partial E}{\partial t} + \frac{1}{2}\frac{\partial^2 E}{\partial A^2}\sigma_A^2\right)dt + \frac{\partial E}{\partial A}\sigma_A dZ$$

$$E\sigma_E = A\sigma_A \frac{\partial E}{\partial A} = A\sigma_A N(d_1).$$

$$E = AN(d_1) - Be^{-rT}N(d_2)$$

$$E\sigma_E = A\sigma_A N(d_1)$$

Through this, the market value and volatility of assets can be determined. In this case, the unknown parameters are estimated using the Broyden method[9].

Based on the estimated market value and volatility of assets, DtD, PD, and expected losses are determined. In particular, DtD and PD are calculated as follows:

$$DtD = \frac{\ln\left(\frac{A}{B}\right) + \left(r - \frac{\sigma_A^2}{2}\right)T}{\sigma_A\sqrt{T}} = d_2$$

$$PD = N(-d_2) = 1 - N(d_2)$$

The DtD represents the number of SD between the expected value of asset and the distress barrier. In cases where the estimated asset values fall below the distress barrier, a rise in the DtD reduces the PD, while a decrease in the number of SD raises the PD.

Based on DtD and PD, expected losses are determined:

$$P = B_T e^{-rT}\big(1 - N(d_2)\big) - A\big(1 - N(d_1)\big)$$

$$d_1 = d_2 + \sigma_A\sqrt{T}.$$

Where,
P – expected losses;
B – distress barrier;

[9] Dennis, J., & Schnabel, R. (1996). Numerical Methods for Unconstrained Optimization and Nonlinear Equations. Society for Industrial and Applied Mathematics Philadelphia.
Broyden method is a numerical iteration algorithm used to solve a system of complex nonlinear equations.

r – risk-free rate;
T – maturity;
d_2 – DtD;
A – market value of assets;
σ_A – volatility of assets.

In the next step, the estimated expected losses at individual bank level are combined into the multivariate distribution at system-wide level.

B. Systemic Contingent Claims Analysis

To assess default risk of the banking system, the expected losses of individual banks are aggregated. In determining the expected losses of the banking system, the marginal distribution of expected losses in banks, the dependence structure and the joint distribution are evaluated.

The marginal distribution of expected losses for each bank is estimated based on the expected losses determined using the Contingent Claims Analysis model, in accordance with Extreme Value Theory (EVT)[10].

[10] Kotz, S., & Nadarajah, S. (1999). Extreme Value Distributions: Theory and Applications. Imperial College Press.

Figure 2. Systemic Contingent Claims Analysis Recursions

Input Estimated Expected Losses	$P^{n}_{E,j}(t), \dots, P^{n}_{E,m}(t)$
↓	
Estimation of Individual Marginal Distributions	$\hat{\mu}_j, \hat{\sigma}_j, \hat{\xi}_j$
↓	
Determination of Dependence Structure	$\Upsilon(\omega) = min\left\{1, max\left\{n\left(\sum_{j=1}^{n} \Lambda_{j=1}^{m} \frac{y_{i,j}}{\omega_j \hat{y}_{.j}}\right)^{-1}, \omega, 1-\omega\right\}\right\}$
↓	
Determination of Joint Distribution	$G_{t,m}(x) = \exp\left\{-\left(\sum_{j=1}^{m} y_{t,j}\right)\Upsilon_t(\omega)\right\}$
↓	
Estimation of Joint Value at Risk $(VaR(a))$	$VaR(a) = inf\{x: F(x) \geq a\}$
↓	
Estimation of Expected Shortfall	$ES(a) = \frac{1}{1-a}\int_{a}^{1} VaR(b)db$

Source: CBU.

The vector-valued series of expected losses with independent and identically distributed random variables is represented as follows:

$$X_j^n = P_{E,j}^n(t), \dots, P_{E,m}^n(t)$$

Where,
j – banks;
$P_{E,m}^n(t)$ – expected losses at time t.

The individual asymptotic tail behavior is modeled in accordance with the Fisher-Tippett-Gnedenko[11] theorem, which determines distribution of normalized maxima (or minima) to be of extremal type:

$$X = \max\left(P_{E,j}^n(t), \dots, P_{E,m}^n(t)\right).$$

When the maximum obtained from n observations are expressed in normalized form using the normalizing constants $\beta_j^n > 0$ and α_j^n, these values converge to the distribution of Generalized Extreme Value (GEV) ($x \in R$):

$$F_n^{[\beta^n x+\alpha^n]}(x) = \lim_{n\to\infty} Pr\left(\frac{X-\alpha^n}{\beta^n} \le y\right) = [F(\beta^n y + \alpha^n)]^n \to H(x).$$

The GEV distribution conforms to one of the three distinct types of extremal behavior: Gumbel (EV_0), Frechet (EV_1) and negative Weibull (EV_2) and is given by the following form:

$$EV_0: H_0(x) = \exp(-\exp(-x)), \qquad if\ x \ge 0,\ \ \xi = 0$$

$$EV_1: H_{1,\xi}(x) = \exp\left(-x^{-\frac{1}{\xi}}\right), \qquad if\ x \in \left[\mu - \frac{\sigma}{\xi}, \infty\right), \xi > 0$$

$$EV_2: H_2(x) = \exp\left(-(-x)^{-\frac{1}{\xi}}\right), \qquad if\ x \in \left(-\infty, \mu - \frac{\sigma}{\xi}\right], \xi < 0.$$

[11] Peon, P. (2020). Statistical Extreme Value Theory. Application to basins of the Basque Country. Master Universitario en Matematicas y Computacion.

By combining the limit distributions, the following probability density function is obtained:

$$b_{\mu,\sigma,\xi}(x) = \frac{1}{\sigma}\left(1+\xi\frac{(x-\mu)}{\sigma}\right)^{\left(-\frac{1}{\xi}\right)-1} \exp\left(-\left(1+\xi\frac{(x-\mu)}{\sigma}\right)_{+}^{-\frac{1}{\xi}}\right)$$

The cumulative distribution function obtained from the probability density function takes the following form:

$$H_{\mu,\sigma,\xi}(x) = \begin{cases} \exp\left(-\left(1+\xi\frac{(x-\mu)}{\sigma}\right)^{-\frac{1}{\xi}}\right), & \text{if } 1+\ \xi\frac{(x-\mu)}{\sigma} \geq 0 \\ \exp(-\exp\left(-\frac{(x-\mu)}{\sigma}\right), & \text{if } x \in\ R,\ \xi = 0 \end{cases}$$

The univariate marginal density function of each expected loss series converging to GEV distribution is defined as:

$$y_i(x) = \left(1+\widehat{\xi_J}\frac{(x-\widehat{\mu_J})}{\widehat{\sigma_J}}\right)_{+}^{-\frac{1}{\xi}} (j = 1, \dots, m).$$

$$1+\widehat{\xi_J}\frac{\left(x-\widehat{\mu_J}\right)}{\widehat{\sigma_J}} > 0$$

$$\sigma_j > 0$$

$$\xi_j \neq 0$$

The mean, variance, skewness and kurtosis of univariate density function are determined

$$\text{Mean: } \begin{cases} \mu + \frac{\sigma(\Gamma(1-\xi)-1)}{\xi},\ \ if\ \xi \neq 0, \xi < 1 \\ \mu + \sigma\gamma,\ if\ \xi = 0 \\ \infty,\ if\ \xi \geq 1 \end{cases}$$

$$\text{Variance: } \begin{cases} \frac{\sigma^2(b_1-b_1^2)}{\xi^2}, if\ \xi \neq 0, \xi < \frac{1}{2} \\ \frac{\sigma^2\pi^2}{6}, if\ \xi = 0 \\ \infty, if\ \xi \geq 1 \end{cases}$$

Skewness: $\begin{cases} \frac{b_3-3b_1b_2+2b_1^3}{(b_2-b_1^2)^{\frac{3}{2}}}, if\ \xi \neq 0 \\ \frac{12\sqrt{6}\zeta(3)}{\pi^3}, if\ \xi = 0 \end{cases}$

Kurtosis: $\begin{cases} \frac{b_4-4b_1b_3+6b_2b_1^2-2b_1^4}{(b_2-b_1^2)^2}, if\ \xi \neq 0 \\ \frac{12}{5}, if\ \xi = 0 \end{cases}$

Where,
$b_p = \Gamma(1 - p\xi), p \in \{1, \dots, 4\}$;
γ – Euler constant (≈0,577)[12];
$\zeta(t)$ – Riemann zeta function[13];
$\Gamma(t)$ – gamma probability density function[14].

These moments are estimated concurrently using numerical iteration[15] via maximum likelihood. In particular, the maximum likelihood estimator in GEV distribution is evaluated by an iteration procedure to maximize the likelihood $\prod_{i=1}^{n} b_{\mu,\sigma,\xi}(x|\theta)$ over the three parameters μ, σ and ξ simultaneously. In the iterative approach, linear combinations of ratios of spacings[16] are used as an initial value of the estimation.

The dependence structure of individual bank expected losses is estimated. In particular, the nonparametric, multivariate dependence function between the marginal distributions of expected losses is defined as:

$$\Upsilon(\omega) = \min\left\{1, \max\left\{n\left(\sum_{j=1}^{n} \Lambda_{j=1}^{m} \frac{y_{i.j}}{\omega_j \widehat{y_{.J}}}\right)^{-1}, \omega, 1-\omega\right\}\right\}$$

$$\widehat{y_{.J}} = \sum_{i=1}^{n} \frac{y_{t,j}}{n}$$

[12] Kaczkowski, S. (2021). On a generalization of Euler's constant. Surveys in Mathematics and its Applications.

[13] Waldschmidt, M. (2017). An introduction to the Riemann zeta function. CIMPA-ICTP Research School.

The Riemann-zeta function is given in the following form: $\zeta(t) = \sum 1\ nt\ \infty\ n$=1 .

[14] Capinski, M., & Kopp, E. (1998). Measure, Integral and Probability. Springer-Verlag.

The gamma function takes the following form: $\Gamma(t) = \int x\ t-1 e\ -xdx\ \infty\ 0$.

[15] Numerical iteration is a method of sequentially computing and updating model parameters to approach optimal values.

[16] In linear combination of ratios of spacings, the difference between the ordered observations is obtained. Logarithmic ratios are compiled by comparing the spacings around the high and low values.

$$0 \leq \max(\omega_1, \dots, \omega_{m-1}) \leq \Upsilon\big(\omega_j\big) \leq 1.$$

Where,
$\Upsilon(\cdot)$ – convex function ($\Upsilon(0) = \Upsilon(1) = 1$);
$\widehat{y_{.j}}$ – reflects the average marginal density of all expected loss values ($i \in n$);
Λ – the smallest ratio of extreme densities across all banks.

By combining the marginal distributions of expected losses across banks, the multivariate extreme value distribution is determined:

$$G_{t,m}(x) = \exp\left\{-\left(\sum_{j=1}^{m} y_{t.j}\right)\Upsilon_t(\omega)\right\}.$$

The probability density function of the multivariate extreme value distribution is expressed as follows:

$$g_{t,m}(x) = \hat{\sigma}_{t,m}^{-1}\left(\left(\sum_{j=1}^{m} y_{t,j}\right)\Upsilon_t(\omega)\right)^{\hat{\xi}_{t,m}+1} \exp\left\{-\left(\sum_{j=1}^{n} y_{t.j}\right)\Upsilon_t(\omega)\right\}.$$

The unknown parameters are estimated by maximizing the following likelihood function:

$$\prod_{j=1}^{m} g_{t,m}(x|\theta)$$

The lognormal likelihood function is determined as the logarithm of the product of probability density functions. In this case, the maximum likelihood estimate of the value θ_0 is evaluated:

$$\sum_{j=1}^{m} lng(x|\theta)$$

$$\hat{\theta}_{MLE} = \underset{\theta\epsilon\Theta}{argmax}\,\hat{l}(\theta|x) \to \theta_0$$

The unknown parameters are determined at the maximum of the log-likelihood function.

The joint expected shortfall is obtained as conditional tail expectation. In particular, the expected shortfall represents the expected value of financial losses on assets that exceed the level of $VaR_{\alpha=0,95}$:

$$ES_a = E[X|X > VaR_a] = \frac{1}{1 - F(VaR_a)} \int_{VaR_a}^{\infty} \big(1 - F(x)\big)dx = \frac{1}{1 - a} \int_a^1 VaR_a \, da$$

$$VaR_a = \inf\{x: F(x) \geq a\}$$

Multivariate expected shortfall at the quantile $(0 < a < 1)$ corresponding to the distribution function F of VaR are calculated as follows:

$$\begin{aligned} ES_{t,\tau,m,a} &= -E\left[z_t \middle| z_t \geq G_{t,\tau,m}^{-1}(a) = VaR_{t,a}\right] \\ &= -\frac{1}{a} \int_0^a G_{t,\tau,m}^{-1}\,(x)dx \end{aligned}$$

$$G^{-1}(a) = \inf\big(x\big|G(x \geq a)\big)$$

Where,
ES – expected shortfall;
$G^{-1}(a)$ – quantile corresponding to probability;
z_t – joint potential losses.

The value of VaR for expected shortfall is determined as follows:

$$VaR_{t,a} = \sup\left\{G_{t,\tau,m}^{-1}(a) | \Pr\left[z_t > G_{t,\tau,m}^{-1}(a)\right] \geq a = 0.95\right\}.$$

In particular, the point estimate of joint potential losses in a certain period is determined:

$$G_{t,\tau,m}^{-1}(a) = \hat{\mu}_{t,m} + \frac{\hat{\sigma}_{t,m}}{\hat{\xi}_{t,m}}\left(\left(-\frac{\ln(a)}{Y_t(\omega)}\right)^{-\hat{\xi}_{t,m}} - 1\right).$$

Expected shortfall is a coherent risk measure[17] defined as a real functional on a space of random variables that satisfy the following axioms:

1) for all X and Y, $v(X + Y) \leq v(X) + v(Y)$;
2) if $X \leq Y$ then, $v(X) \leq v(Y)$;
3) for positive constant λ, $v(\lambda X) = \lambda v(X)$;
4) for constant c, $v(X + c) = v(X) + c$.

As the final outcome of the Systemic Conditional Claims Analysis model, the value of multivariate expected shortfall is considered. In conducting Systemic Conditional Claims Analysis for the banking system, banks' systemic importance, stock exchange listing, and the availability of required time-series data are taken into account.

[17] Lam, T., Verma, A., Low, B., & Jaillet, P. (2023). Risk-Aware Reinforcement Learning with Coherent Risk Measures and Non-linear Function Approximation.