Growth at Risk Model

Abstract

This methodological framework was developed to estimate the probability of a sharp economic downturn under adverse scenarios, taking into account the effects of current macro-financial conditions on future real economic growth. The framework focuses on a model for assessing economic growth under risk conditions. The model consists of two main stages: estimating the relationship between economic growth and its determinants using quantile regression, followed by fitting a skewed-t distribution to the resulting conditional growth estimates.

Keywords: economic growth, quantile, financial cycle index, relative error, regression, random variable, distribution function.

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Growth at Risk Model

The Growth at Risk (GaR) [1] model links current macrofinancial conditions with future real GDP growth. The main advantage of this analysis is that it estimates the entire distribution of future real GDP growth. Using the GaR model, the likelihood of adverse scenarios can be quantified.

The evolution of changes in the financial system's condition may allow the identification of periods of change for future economic activity. In particular, the GaR model assesses the probability of a sharp economic downturn. In this context, the distribution of future real GDP growth is estimated based on data on macrofinancial variables.

The GaR model is implemented in two stages. In the first stage, the FCI[2] and current real GDP growth are taken as macrofinancial variables. These variables are utilized to estimate future real GDP growth at different quantiles using the quantile regression model. In the second stage, after being estimated by the quantile regression model, a skew-t[3] distribution function is fitted to the estimated discrete values of the conditional quantile regressions in order to estimate the conditional continuous distribution of future real GDP growth.

Quantile regression estimation

The first stage of the GaR model involves estimating the quantile regression model. In particular, the relationship between the current FCI and current real GDP growth with future real GDP growth is estimated using the quantile regression model. The quantile regression is estimated at different quantiles $q \in (10, 25, 50, 75, 90)$ and takes the following form:

$$y_{t+h}^{q} = \alpha^{q} + \beta_{\mathrm{C}}^{q} X_{\mathrm{C},t} + \beta_{G}^{q} X_{G,t} + \varepsilon_{t+h}^{q}$$

Where,

q – quantile;

[1] Prasad, A., Elekdag, S., Jeasakul, P., Lafarguette, R., Alter, A., Xiaochen Feng, A., & Wang, Ch. (2019). Growth at Risk: Concept and Application in IMF Country Surveillance. IMF Working Paper.

[2] The FCI methodology is presented in the Methodological Framework of the Central Bank.

[3] Basalamah, D. (2017). Statistical Inference for a New Class of Skew T Distribution and Its Related Properties. Bowling Green State University.

t – time series;
y^q_{t+h} – real GDP growth h quarters ahead for quantile q;
$X_{\mathrm{C},t}$ – financial cycle index;
$X_{G,t}$ – current real GDP growth;
α^q – constant term;
ε^q_{t+h} – residual;
$\beta^q_{\mathrm{C}}, \beta^q_G$ – unknown coefficients.

Estimated discrete values of quantile regression for the next period are determined using quantile regression parameters estimated at different quantiles. The conditional continuous distribution function is obtained by fitting the t-skew distribution to these discrete values..

Deriving the conditional continuous distribution of future growth

After calculating various quantiles of the future period's real GDP growth, the entire conditional distribution can also be computed. The second stage of the GaR model involves creating a continuous distribution of the conditional growth of future real GDP. The conditional distribution of future real GDP growth is derived by fitting the Azzalini and Capitanio skew-t distribution to the predicted values of the conditional quantile regressions[4].

Azzalini and Capitanio's skew-t distribution is characterized by parameters for mean, standard deviation, skewness and degree of freedom. The skew-t distribution is derived using the Studentt distribution[5]. In this case, the standard skew-t distribution

[4] Drenkovska, M., & Volcjak, R. (2022). Growth-at-Risk and Financial Stability: Concept and Application for Slovenia. Banka Slovenije.

[5] The probability density function of the Student t-distribution has the following form:

$$t_v(x) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{\pi v}\,\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{x^2}{v}\right)^{-\frac{v+1}{2}}$$
$$v = t-1$$
$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx.$$

Where,

$t_v(x)$ – probability density function;

v – degrees of freedom;

t – sample size;

$\Gamma(p)$ – Gamma function.

with zero mean and one standard deviation has an asymmetric form, and the probability density function has the following form:

$$f_{v,\beta}(x) = 2t_v(x)T_{v+1}\left(\beta x\sqrt{\frac{v+1}{x^2+v}}\right)$$

Where,

β – shape parameter;

v – degrees of freedom;

$t_v(x)$ – (v) the probability density function of the Student-t distribution with degrees of freedom;

T_{v+1} – distribution function of the Student-t distribution with $(v+1)$ degrees of freedom.

The probability density function of a random variable with a general z skew-t distribution with mean ξ and standard deviation ω is expressed as follows:

$$f_{v,\beta}\left(\frac{x-\xi}{\omega}\right) = 2t_v\left(\frac{x-\xi}{\omega}\right)T_{v+1}\left(\beta\frac{x-\xi}{\omega}\sqrt{\frac{v+1}{\left(\frac{x-\xi}{\omega}\right)^2+v}}\right)$$

$$z = \frac{x-\xi}{\omega}$$

The probability density function of the skew-t distribution is obtained by multiplying the probability density function of the Student-t distribution with the distribution function of the Student-t distribution. To demonstrate that the function $f_{v,\beta}(x)$ is a probability density function[6], the function is first considered in its general form.

[6] The condition for being the probability density function is: $\int_{-\infty}^{\infty} f(x)dx = 1$.

If X is a random variable, the probability density function f(x) is symmetric with respect to the ordinate, and G(x) is an arbitrary distribution function, then for any α, $f(x;\alpha) = 2f(x)G(\alpha x)$ is a probability density function as follows[7]:

$$\int_{-\infty}^{\infty} f(x;\alpha)dx = \int_{-\infty}^{\infty} 2f(x)G(\alpha x)dx$$

$$= \int_{-\infty}^{\infty} f(x)G(\alpha x)dx + \int_{-\infty}^{\infty} f(x)G(\alpha x)dx$$

$$= \int_{-\infty}^{\infty} f(x)G(\alpha x)dx + \int_{-\infty}^{\infty} f(x)(1 - G(-\alpha x))dx$$

$$= \int_{-\infty}^{\infty} f(x)G(\alpha x)dx + \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{\infty} f(x)G(-\alpha x)dx$$

$$= \int_{-\infty}^{\infty} f(x)G(\alpha x)dx + \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{\infty} f(-x)G(\alpha x)dx$$

$$= \int_{-\infty}^{\infty} f(x)G(\alpha x)dx + \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{\infty} f(x)G(\alpha x)dx$$

$$= \int_{-\infty}^{\infty} f(x)dx = 1.$$

In the process of proving this, the symmetry property of the Student-t distribution's probability density function, $f(x) = f(-x)$, and the property of distribution function, $F(-\alpha x) = 1 - F(\alpha x)$, are used. Since this proof is established for functions in general forn, it is also applicable to the skew-t distribution. In particular, the standard skew-t distribution is composed of the product of the probability density function of the symmetric Student-t distribution $t_v(x)$ and the distribution function of the Student-t distribution $T_{v+1}\left(\beta x\sqrt{\frac{v+1}{x^2+v}}\right)$. The process, which is valid for the standard skew-t distribution, is also applicable to the general skew-t distribution. The optimal estimate of the unknown parameters is obtained by minimizing the squared

[7] Berlik, S. (2006). Directed Evolutionary Algorithms. Dortmund University.

distance between the estimated quantile regression values and the inverse distribution function of the skew-t distribution:

$$\{\hat{\xi}, \hat{\omega}, \hat{\alpha}, \hat{v}\} = argmin \sum_{q} \left(\hat{Q}_{y_{t+h|x_t}}(q|x_t) - F^{-1}(\xi, \omega, \alpha, v) \right)^2$$

Where,

$\hat{Q}_{y_{t+h|x_t}}(q|x_t)$ – values estimated using quantile regression;

$F^{-1}(\xi, \omega, \alpha, v)$ – the inverse distribution function of the skew-t distribution.

The skew-t distribution can be fitted by finding four parameters that minimize the squared distance. In this case, it is necessary to have the inverse distribution function of the skew-t distribution expressed as a function of the parameters in order to solve the minimization problem. The inverse distribution function of the skew-t distribution can be determined using the Cornish-Fisher expansion[8]. In particular, the moment-based estimation of the Cornish-Fisher expansion has the following form:

$$\begin{aligned}
F^{-1}(x) \approx\ & z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)E(x^3) + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)E(x^4) \\
& - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)E(x^3)^2 + \frac{1}{120}(z_\alpha^4 - 6z_\alpha^2 + 3)E(x^5) \\
& - \frac{1}{24}(z_\alpha^4 - 5z_\alpha^2 + 2)E(x^3)E(x^4) + \frac{1}{324}(12z_\alpha^4 - 53z_\alpha^2 + 17)E(x^3)^3 \\
& + \frac{1}{720}(z_\alpha^5 - 10z_\alpha^3 + 15z_\alpha)E(x^6) - \frac{1}{180}(2z_\alpha^5 - 17z_\alpha^3 + 21z_\alpha)E(x^3)E(x^5) \\
& - \frac{1}{384}(3z_\alpha^5 - 24z_\alpha^3 + 29z_\alpha)E(x^4)^2 \\
& + \frac{1}{288}(14z_\alpha^5 - 103z_\alpha^3 + 107z_\alpha)E(x^3)^2E(x^4) \\
& - \frac{1}{7776}(252z_\alpha^5 - 1688z_\alpha^3 + 1511z_\alpha)E(x^3)^4 + \cdots
\end{aligned}$$

[8] Amedee-Manesme, C., & Barthelemy, F. (2012). Cornish-Fisher Expansion for Real Estate Value at Risk. CergyPontoise University.

Where,

z_α – the inverse distribution function of the standard normal distribution;

$E(x^n)$ – moments of order n.

When the first four moments of the distribution are used, the following approximate equation is obtained:

$$F^{-1}(x) \approx z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)E(x^3) + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)E(x^4)$$
$$-\frac{1}{36}(2z_\alpha^3 - 5z_\alpha)E(x^3)^2.$$

The relationship between the general skew-t distribution and the standard skew-t distribution takes the following form[9]:

$$Y_{\xi,\omega,\alpha,v} = \xi + \omega Y_{0,1,\alpha,v}$$

Where,

$Y_{0,1,\alpha,v}$ – standard skew-t distribution;

$Y_{\xi,\omega,\alpha,v}$ – general skew-t distribution.

In determining the variance of the standard skew-t distribution, the following notations are introduced:

$$b_v = \sqrt{\frac{v}{\pi}}\frac{\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}$$

$$\delta_\alpha = \frac{\alpha}{\sqrt{1+\alpha^2}}.$$

Following these notations, the skewness parameter α which varies over $(-\infty; +\infty)$, becomes one that varies over $(-1; +1)$. After that, the variance of the standard skew-t distribution takes the following form:

[9] Uthaisaad, Ch., & Martin, D. (2018). The Azzalini Skew-t Information Matrix Evaluation and Use for Standard Error Calculations. SSRN.

$$\sigma_z^2 = var(z) = \frac{v}{v-2} - (b_v\delta_\alpha)^2$$

The parameter vector of the skew-t distribution (ξ, ω, α, v) is transformed into the vector of the usual mean, standard deviation, skewness, and kurtosis $(\mu, \sigma, \gamma, \kappa)$ based on the following equations:

$$\mu = \xi + \omega b_v \delta_\alpha$$

$$\sigma^2 = \omega^2 \sigma_z^2, \qquad v > 2$$

$$\gamma = \frac{b_v\delta_\alpha}{\sigma^3}\left[\frac{v(3-\delta_\alpha^2)}{v-3} - \frac{3v}{v-2} + 2(b_v\delta_\alpha)^2\right], \qquad v > 3$$

$$\kappa = \frac{1}{\sigma_z^4}\left[\frac{3v^2}{(v-2)(v-4)} - \frac{4(b_v\delta_\alpha)^2 v(3-\delta_\alpha^2)}{v-3} + \frac{6v(b_v\delta_\alpha)^2}{v-2} - 3(b_v\delta_\alpha)^4\right] - 3,$$

$$v > 4.$$

Using notations, the Cornish-Fisher expansion is obtained and the approximate inverse distribution function of skew-t distribution is derived. The unknown parameters of the skew-t distribution are determined by applying the approximate inverse skew-t distribution in a minimization problem. The one-percent probability value of the fitted skew-t distribution is determined and assigned as the value of the GaR model.

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