Exponential Moving Average

Abstract

This methodological framework is based on the exponential moving average (EMA) method, which assigns greater weight to recent observations than the simple moving average, thereby capturing current changes more effectively while reducing short-term fluctuations in the analysed indicators. The framework describes the application of the EMA method to smooth indicator values by mitigating the effects of random and transitory fluctuations. In particular, the optimal smoothing constant is determined based on approaches that consider the average age of observations in the moving average, as well as the mathematical expectation and variance of the difference between the moving average and the underlying stationary time series.

Keywords: variance, mathematical expectation, smoothing constant, stationary time series, mean, exponential.

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Exponential Moving Average

To determine the moving average of stationary time series[1], simple and exponential moving average methods are widely used[2] in practice. Each observed time series is considered stationary when its mean is constant and its error term can be represented by random fluctuations. In particular, a stationary time series is expressed as follows:

$$D_t = \mu + \epsilon_t.$$

Where,
D_t – stationary time series;
μ – the mean value of a stationary time series;
ϵ_t – random error drawn from a normal distribution with mean zero and variance (σ^2).

Simple moving average

The simple moving average of order N is calculated as the arithmetic mean of the most recent N observations of the stationary time series and, for period t is obtained as follows:

$$F_t = \frac{1}{N}\sum_{i=t-N+1}^{t} D_i = \frac{1}{N}(D_t + D_{t-1} + \cdots + D_{t-N+1})$$

Where,
F_t – simple moving average;
D_t – stationary time series;
N – order of the moving average.

In calculating the moving average, each time a new observation of the stationary time series is added, the arithmetic mean of the last N observations must be recalculated. To avoid this recalculation, F_{t-1} can be written in the following form:

[1] Camps, E. C. (2022). Introduction to Time Series and Forecasting.
Stationary time series are time series whose mean and standard deviation remain constant across different time intervals.
[2] Steven, N., & Tava, L. (2015). Production and operation analysis.

$$F_{t-1} = \frac{1}{N}\sum\nolimits_{i=t-N}^{t-1} D_i = \frac{1}{N}[D_{t-1} + D_{t-2} + \cdots + D_{t-N+1} + D_{t-N}]$$
$$\frac{1}{N}[D_{t-1} + D_{t-2} + \cdots + D_{t-N+1} + D_{t-N} + D_t - D_t] = F_t + \frac{1}{N}[D_{t-N} - D_t]$$
$$F_t = F_{t-1} + \frac{1}{N}[D_t - D_{t-N}].$$

Since all indicators in a stationary time series are assigned the equal weights of $\frac{1}{N}$, the moving average will lag significantly behind the actual observed values. To address this issue, the exponential moving average is used.

Exponential moving average

The exponential moving average (F_t) assigns greater weight to the recent values of the stationary time series and is expressed as follows:

$$F_t = \alpha D_t + (1 - \alpha)F_{t-1}.$$

Where,
F_t – the value of the exponential moving average at period t;
F_{t-1} – the value of the exponential moving average at period $t - 1$;
D_t – stationary time series;
α – a weight between zero and one ($0 < \alpha \leq 1$) placed to the current observation as a smoothing constant.

Using the above formula, F_{t-1} is determined as follows:

$$F_{t-1} = \alpha D_{t-1} + (1 - \alpha)F_{t-2}.$$

Where,
F_{t-1} – the value of the exponential moving average at period $t - 1$;
F_{t-2} – the value of the exponential moving average at period $t - 2$;
D_{t-1} – stationary time series in period $t - 1$;
α – a weight between zero and one ($0 < \alpha \leq 1$) assigned to the current observation as a smoothing constant.

By substitution, the following equation is obtained:

$$F_t = \alpha D_t + \alpha(1-\alpha)D_{t-1} + (1-\alpha)^2 F_{t-2}.$$

In this way, the infinite expansion for F_t is obtained as follows:

$$F_t = \sum_{i=0}^{\infty} \alpha\,(1-\alpha)^i D_{t-i} = \sum_{i=0}^{\infty} a_i\, D_{t-i}.$$

a_i is the weight given to each observation, and their sum is equal to one:

$$a_0 > a_1 > a_2 > \cdots > a_i = \alpha(1-\alpha)^i$$

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} \alpha(1-\alpha)^i = \alpha(1-\alpha)^0 + \alpha(1-\alpha)^1 + \alpha(1-\alpha)^2 + \cdots$$

$$= \alpha(1 + (1-\alpha)^1 + (1-\alpha)^2 + (1-\alpha)^3 + \cdots)$$

$$= \alpha * \frac{1}{1-(1-\alpha)} = 1.$$

The exponential moving average at each step requires the value of the previous period's exponential moving average. Therefore, in the initial calculations, the arithmetic mean of the initial period is used.

In the exponential moving average, a set of declining weights is applied to all past indicators of the stationary time series.

If the value of α is large, more weight is assigned to the current observation and less weight to past observations. If α is small, greater weight is given to past data, causing the influence of older observations to persist longer and reducing the responsiveness to recent changes. For this reason, determining the optimal value of α is considered important.

The optimal value of the smoothing constant α is determined using two methods. In the first method, the average age (A) of the moving averages is equalized. In the second method, the mathematical expectation and standard deviation of the difference $(F_t - D_t)$ between the moving averages and the stationary time series are equalized, respectively.

Equalization of average age (A)

The average age (A) of a stationary time series is determined as the sum of the products of each observation's position and its weight. In the simple moving average, equal weights of $\frac{1}{N}$ are assigned to the last N observations, and the average age (A) is calculated as follows:

$$\mathrm{A} = \frac{1}{N}(1 + 2 + 3 + \ldots + N) = \left(\frac{1}{N}\right)\frac{N * (N+1)}{2} = \frac{N+1}{2}.$$

Where,
A – average age of stationary time series;
N – the number of observations.

In the exponential moving average, the weight applied to the data at period i is equal to $\alpha(1-\alpha)^{i-1}$. For a time series with infinite historical observations, the average age of the exponential moving average is determined as follows:

$$\mathrm{A} = \sum_{i=1}^{\infty} i\alpha\,(1-\alpha)^{i-1} = 1\alpha(1-\alpha)^0 + 2\alpha(1-\alpha)^1 + 3\alpha(1-\alpha)^2 \ldots$$

$$\mathrm{A}(1-\alpha) = \sum_{i=1}^{\infty} i\alpha\,(1-\alpha)^{i} = 1\alpha(1-\alpha)^1 + 2\alpha(1-\alpha)^2 + 3\alpha(1-\alpha)^3 \ldots$$

$$\mathrm{A} - A(1-\alpha) = \alpha(1-\alpha)^0 + \alpha(1-\alpha)^1 + \alpha(1-\alpha)^2 \ldots$$

$$= \alpha\frac{1}{1-(1-\alpha)} = 1$$

$$\mathrm{A} = \frac{1}{\alpha}$$

Where,
A – average age of stationary time series;
α – a weight between zero and one ($0 < \alpha \leq 1$) placed on the current observation as a smoothing constant.

By equating the average age of the moving averages, the following result is obtained:

$$\frac{\mathrm{N}+1}{2}=\frac{1}{\alpha}$$

$$\alpha=\frac{2}{N+1}$$

Where,

N – order of the moving average;
α – a weight between zero and one ($0 < \alpha \leq 1$) placed on the current observation as a smoothing constant.

Equalization of mathematical expectations and variances

In this process, the mathematical expectation and standard deviation of the difference between the moving averages and the stationary time series $(F_t - D_t)$ are respectively equalized.

A simple moving average is calculated as follows:

$$F_t = \frac{1}{N}\sum_{i=t-N+1}^{t} D_i.$$

Where,

F_t – simple moving average;
D_i – stationary time series;
N – order of the moving average.

The mathematical expectation of the difference $(F_t - D_t)$ between the simple moving average and the stationary time series is determined as follows:

$$E(F_t - D_t) = E(F_t) - E(D_t) = \frac{1}{N}\sum_{i=t-N+1}^{t} E(D_i) - E(D_t)$$

$$= \frac{1}{N}(N\mu) - \mu = 0.$$

The variance of the difference $(F_t - D_t)$ between the simple moving average and the stationary time series is calculated as follows:

$$\mathrm{Var}(F_t - D_t) = Var(F_t) + Var(D_t) = Var\left(\frac{1}{N}\sum_{i=t-N+1}^{t} D_i\right) + Var(D_t)$$
$$= \frac{1}{N^2}(N\sigma^2) + \sigma^2 = \sigma^2\left(\frac{1}{N} + 1\right) = \sigma^2\left(\frac{N+1}{N}\right).$$

Using the defined variance, the standard deviation is calculated as follows:

$$\sigma_e = \sigma\sqrt{\frac{N+1}{N}}.$$

In this manner, the mathematical expectation and the standard deviation of the difference between the exponential moving average and the stationary time series are calculated as follows:

$$F_t = \alpha D_t + \alpha(1-\alpha)^1 D_{t-1} + \alpha(1-\alpha)^2 D_{t-2} + \cdots$$
$$E(F_t) = E(\alpha D_t + \alpha(1-\alpha)^1 D_{t-1} + \alpha(1-\alpha)^2 D_{t-2} + \cdots)$$
$$= \alpha E(D_t) + \alpha(1-\alpha)^1 E(D_{t-1}) + \alpha(1-\alpha)^2 E(D_{t-2}) + \cdots$$
$$= \alpha\mu + \alpha(1-\alpha)^1\mu + \alpha(1-\alpha)^2\mu + \cdots$$
$$= \alpha\mu\frac{1}{1-(1-\alpha)} = \mu$$
$$E(F_t - D_t) = E(F_t) - E(D_t) = \mu - \mu = 0$$

$$Var(F_t) = Var(\alpha D_t + \alpha(1-\alpha)^1 D_{t-1} + \alpha(1-\alpha)^2 D_{t-2} + \cdots)$$
$$= \alpha^2\sigma^2 + (1-\alpha)^2\alpha^2\sigma^2 + (1-\alpha)^4\alpha^2\sigma^2 + \cdots$$
$$= \alpha^2\sigma^2(1 + (1-\alpha)^2 + (1-\alpha)^4 + \cdots)$$
$$= \alpha^2\sigma^2\frac{1}{1-(1-\alpha)^2} = \frac{\sigma^2\alpha}{2-\alpha}$$
$$\mathrm{Var}(F_t - D_t) = Var(F_t) + Var(D_t) = \frac{\sigma^2\alpha}{2-\alpha} + \sigma^2 = \sigma^2\frac{2}{2-\alpha}$$

$$\sigma_e = \sigma\sqrt{\frac{2}{2-\alpha}}.$$

For both the moving average and the stationary time series, the mathematical expectation of their differences is equal to zero. By equating the standard deviations, the optimal value of α is determined as follows:

$$\sigma\sqrt{\frac{N+1}{N}} = \sigma\sqrt{\frac{2}{2-\alpha}}$$

$$\frac{N+1}{N} = \frac{2}{2-\alpha}$$

$$\alpha = \frac{2}{\mathrm{N}+1}$$

Where,

N – order of the moving average;

α – a weight between zero and one ($0 < \alpha \leq 1$) placed on the current observation as a smoothing constant.

The results were the same for both methods of determining the optimal value of the smoothing constant. This indicates that the average age of both moving averages, as well as the mathematical expectation and the standard deviation of the differences between the moving averages and the stationary time series, are the same. Moreover, although these methods are intended for stationary time series, because of the fact that time series are not always stationary, the standard deviations of both moving averages will be higher than the standard deviation of the observed time series.