Kubik splayn interpolyatsiyasi
(Cubic spline interpolation)

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Annotatsiya

Ushbu maqolada ma’lum nuqtalar to‘plamidan o‘tadigan uzluksiz hosilalarga ega uchinchi darajali uzluksiz egri chiziq funksiyasini hosil qilish usuli yoritilgan. Kubik splayn interpolyatsiyasida egri chiziq funksiyasining noma’lum koeffitsiyentlarini aniqlashda barcha ma’lumotlar nuqtalarida interpolyatsiyalanganligi, shuningdek ushbu funksiyaning birinchi va ikkinchi tartibli hosilalari ma’lum oraliqlarda uzluksizligi xossalaridan foydalaniladi. Mazkur maqola kubik splayn interpolyatsiyasi orqali yillik yoki choraklik ko‘rinishdagi ma’lumotlarni yuqoriroq aniqlikda oylik yoki choraklik ko‘rinishga o‘tkazish tizimini takomillashtirish maqsadida tayyorlangan.

Tayanch so‘zlar: egri chiziq funksiyasi, hosila, interpolyatsiya, kubik splayn, matritsa.

Kubik splayn interpolyatsiyasi (Cubic spline interpolation)

Kubik splayn interpolyatsiyasi[1] bir nechta nuqtalar orqali o'tadigan uchinchi darajali uzluksiz egri chiziq funksiyasini hosil qilish usuli hisoblanadi. Ushbu usul orqali aniqlangan uzluksiz egri chiziq funksiyasi nuqtalar orasidagi ma'lumotlarni aniqroq ifodalashga xizmat qiladi.

Kubik splayn interpolyatsiyasi metodologiyasiga[2] ko'ra bir nechta nuqtalar orqali o'tadigan uzluksiz egri chiziq funksiyasi quyidagicha ifodalanadi:

$$S(x) = \begin{cases} s_1(x)\ agar\ x_1 \le x \le x_2 \\ s_2(x)\ agar\ x_2 \le x \le x_3 \\ \vdots \\ s_{n-1}(x)\ agar\ x_{n-1} \le x \le x_n \end{cases}$$

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i$$
$$i = 1, 2, \dots, n-1$$

Bu yerda, s_i – uchinchi darajali egri chiziq funksiyasi, n – nuqtalar soni, x_n – nuqtalar joylashgan davrlar, a_i, b_i, c_i va d_i – egri chiziq funksiyasining noma'lum koeffitsiyentlari.

Yuqoridagi uchinchi darajali egri chiziq funksiyasidan birinchi va ikkinchi tartibli hosilalar olinadi:

$$s_i'(x) = 3a_i(x - x_i)^2 + 2b_i(x - x_i) + c_i$$
$$s_i''(x) = 6a_i(x - x_i) + 2b_i$$
$$i = 1, 2, \dots, n-1$$

Shuningdek, funksiya tarkibidagi noma'lum koeffitsiyentlarini aniqlashda kubik splayn interpolyatsiyasining $S(x)$ funksiyasi barcha ma'lumotlar nuqtalarida interpolyatsiyalanganligi, shuningdek $S(x)$ funksiyasi, ushbu funksiyaning birinchi va ikkinchi tartibli hosilalari $[x_1, x_n]$ oraliqda uzluksizligi xossalaridan foydalaniladi.

[1] Berilgan nuqtalar to'plamidan o'tadigan uzluksiz hosilalarga ega uchinchi darajali uzluksiz egri chiziq.

[2] McKinley, S., & Levine, M. (1998). Cubic Spline Interpolation.

$S(x)$ funksiyasi barcha ma’lumotlar nuqtalarida interpolyatsiyalashi xususiyatidan kelib chiqqan holda, quyidagi tenglik o‘rinli bo‘ladi:

$$s_i(x_i) = a_i(x_i - x_i)^3 + b_i(x_i - x_i)^2 + c_i(x_i - x_i) + d_i$$
$$s_i(x_i) = d_i$$

Egri chiziq funksiyasi butun oraliq bo‘ylab uzluksiz bo‘lish xossasiga ko‘ra, har bir subfunksiyalar ma’lum nuqtalarida birlashadi:

$$s_i(x_i) = s_{i-1}(x_i)$$
$$i = 2, \dots, n$$
$$s_{i-1}(x_i) = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1}$$
$$d_i = a_{i-1}(x_i - x_{i-1})^3 + b_{i-1}(x_i - x_{i-1})^2 + c_{i-1}(x_i - x_{i-1}) + d_{i-1}$$
$$i = 2, \dots, n-1$$
$$h = x_i - x_{i-1}$$
$$d_i = a_{i-1}h^3 + b_{i-1}h^2 + c_{i-1}h + d_{i-1}$$

$S(x)$ funksiyasining birinchi tartibli hosilasi $[x_1, x_n]$ oralig‘ida uzluksiz bo‘lish xossasiga ko‘ra, quyidagi tengliklar hosil bo‘ladi:

$$s'_i(x_i) = s'_{i-1}(x_i)$$
$$s'_i(x_i) = 3a_i(x_i - x_i)^2 + 2b_i(x_i - x_i) + c_i$$
$$s'_i(x_i) = s'_{i-1}(x_i) = c_i$$
$$c_i = 3a_{i-1}(x_i - x_{i-1})^2 + 2b_{i-1}(x_i - x_{i-1}) + c_{i-1}$$
$$c_i = 3a_{i-1}h^2 + 2b_{i-1}h + c_{i-1}$$

$S(x)$ funksiyasining ikkinchi tartibli hosilasi $[x_1, x_n]$ oralig‘ida uzluksiz bo‘lish xossasidan kelib chiqqan holda, quyidagicha aniqlanadi:

$$s''_i(x_i) = s''_{i-1}(x_i)$$
$$s''_i(x_{i+1}) = s''_{i+1}(x_{i+1})$$
$$s''_i(x_i) = 6a_i(x_i - x_i) + 2b_i$$
$$s''_i(x_i) = s''_{i-1}(x_i) = 2b_i$$

$$2b_i = 6a_{i-1}(x_i - x_{i-1}) + 2b_{i-1}$$
$$2b_i = 6a_{i-1}h + 2b_{i-1}$$
$$2b_{i+1} = 6a_i h + 2b_i$$

$s_i''(x_i) = M_i$ belgilash kiritilib, yuqoridagi ifodalar soddalashtiriladi:

$$s_i''(x_i) = 2b_i$$
$$M_i = 2b_i$$
$$b_i = \frac{M_i}{2}$$

d_i qiymati $S(x)$ funksiyaning x_i nuqtadagi qiymatiga teng bo‘ladi ($s_i(x_i) = d_i$). a_i qiymati esa, yuqoridagi belgilashlar orqali qaytadan ifodalanadi:

$$2b_{i+1} = 6a_i h + 2b_i$$
$$6a_i h = 2b_{i+1} - 2b_i$$
$$a_i = \frac{2b_{i+1} - 2b_i}{6h}$$
$$a_i = \frac{2\left(\frac{M_{i+1}}{2}\right) - 2\left(\frac{M_i}{2}\right)}{6h}$$
$$a_i = \frac{M_{i+1} - M_i}{6h}$$

c_i koeffitsiyent qiymati kiritilgan belgilashlar orqali qaytadan ifodalanadi:

$$d_{i+1} = a_i h^3 + b_i h^2 + c_i h + d_i$$
$$c_i h = d_{i+1} - a_i h^3 - b_i h^2 - d_i$$
$$c_i = \frac{d_{i+1} - a_i h^3 - b_i h^2 - d_i}{h}$$
$$c_i = (-a_i h^2 - b_i h) - \frac{d_i - d_{i+1}}{h}$$
$$c_i = \frac{d_{i+1} - d_i}{h} - (\frac{M_{i+1} - M_i}{6h} h^2 - \frac{M_i}{2} h)$$
$$c_i = \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} + 2M_i}{6}\right) h$$

Aniqlangan tengliklar orqali uzluksiz egri chiziq funksiyasining noma’lum koeffitsiyentlarini hisoblash uchun quyidagi ifodalar hosil bo‘ladi:

$$\begin{cases} a_i = \dfrac{M_{i+1} - M_i}{6h} \\ b_i = \dfrac{M_i}{2} \\ c_i = \dfrac{y_{i+1} - y_i}{h} - \left(\dfrac{M_{i+1} + 2M_i}{6}\right) h \\ d_i = y_i \end{cases}$$

Uzluksiz egri chiziq funksiyasining ikkinchi tartibli hosilasini ($s_i''(x_i) = M_i$) aniqlash ushbu egri chiziq funksiyasining noma’lum koeffitsiyentlarini hisoblash uchun yetarli bo‘ladi. Bunda, noma’lum qiymatlarni aniqlashni birmuncha osonlashtirish maqsadida ifodalar matritsa ko‘rinishiga keltiriladi:

$$c_{i+1} = 3a_i h^2 + 2b_i h + c_i$$

$$3\left(\frac{M_{i+1} - M_i}{6h}\right) h^2 + 2\left(\frac{M_i}{2}\right) h + \frac{y_{i+1} - y_i}{h} - \left(\frac{M_{i+1} + 2M_i}{6}\right) h$$
$$= \frac{y_{i+2} - y_{i+1}}{h} - \left(\frac{M_{i+2} + 2M_{i+1}}{6}\right) h$$
$$h\left(\frac{3M_{i+1} - 3M_i}{6} + \frac{6M_i}{6} - \left(\frac{M_{i+1} + 2M_i}{6}\right) + \left(\frac{M_{i+2} + 2M_{i+1}}{6}\right)\right) = \frac{y_i - 2y_{i+1} + y_{i+2}}{h}$$
$$\frac{h}{6} * (M_i + 4M_{i+1} + M_{i+2}) = \frac{y_i - 2y_{i+1} + y_{i+2}}{h}$$
$$M_i + 4M_{i+1} + M_{i+2} = 6\left(\frac{y_i - 2y_{i+1} + y_{i+2}}{h^2}\right)$$
$$i = 1,2,3, \dots, n-1$$

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Hosil bo‘lgan matritsa tenglamasida n ta noma’lum $(M_1, M_2, \dots, M_{n-1}, M_n)$ hamda $n-2$ ta tenglik mavjud. Bu esa, noma’lumlarni to‘liq aniqlash imkonini bermaydi. Shu sababli, noma’lum qiymatlar kubik splayn interpolyatsiyasining boshlang‘ich va oxirgi chegaradagi subfunksiyalari uchun tabiiy splayn (natural splines), parabolik chegaraviy splayn (parabolic runout spline) va kubik chegaraviy splayn (cubic runout spline) shartlarini kiritish orqali aniqlanadi.

Tabiiy splayn (natural splines) sharti $S(x)$ funksiyaning ikkinchi tartibli hosilasi $(s_i''(x_i) = M_i)$ birinchi va oxirgi nuqtalarida nolga teng bo‘lishini $(M_1 = M_n = 0)$ o‘z ichiga oladi. Buning natijasida, yuqoridagi matritsa tenglamasi quyidagi ko‘rinishga keladi:

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \\ 0 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Matritsani soddalashtirish maqsadida M_1 va M_n nolga teng bo‘lganligi sababli ularning qiymatlariga mos keluvchi dastlabki matritsaning birinchi va oxirgi ustunlari olib tashlanadi.

$$\begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Hosil bo‘lgan matritsadan ko‘rish mumkinki $n-2$ ta noma’lum $(M_2, M_3, \dots, M_{n-2}, M_{n-1})$ hamda $n-2$ ta tenglik mavjud, natijada M_2 dan M_{n-1} gacha bo‘lgan noma’lum qiymatlar aniqlanadi. Aniqlangan M_i qiymatlari asosida uzluksiz egri chiziq funksiyasi noma’lum koeffitsiyentlari hisoblanib, bir nechta nuqtalar orqali o‘tadigan uzluksiz egri chiziq funksiyasi aniqlanadi.

Parabolik chegaraviy splayn (parabolic runout spline) sharti $S(x)$ funksiyani ikkinchi tartibli hosilasining $(s_i''(x_i) = M_i)$ birinchi va oxirgi nuqtalari, mos ravishda ikkinchi va oxiridan bitta oldingi nuqtalariga teng bo‘lishini $(M_1 = M_2\ va\ M_n = M_{n-1})$ o‘z ichiga oladi. Natijada, yuqoridagi dastlabki matritsa tenglamasi quyidagi ko‘rinishiga keladi:

$$\begin{bmatrix} 5 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Matritsa tenglamasini yechish orqali M_2 dan M_{n-1} gacha bo‘lgan noma’lum qiymatlar aniqlanadi. Aniqlangan M_i qiymatlari asosida uzluksiz egri chiziq funksiyasi noma’lum koeffitsiyentlari hisoblanib, bir nechta nuqtalar orqali o‘tadigan uzluksiz egri chiziq funksiyasi aniqlanadi.

Kubik chegaraviy splayn (cubic runout spline) sharti $S(x)$ funksiyani ikkinchi tartibli hosilasining $(s_i''(x_i) = M_i)$ ma’lum munosabatlarini $(M_1 = 2M_2 - M_3\ va\ M_n = 2M_{n-1} - M_{n-2})$ o‘z ichiga oladi va yuqoridagi matritsa tenglamasi quyidagi ko‘rinishiga keladi:

$$\begin{bmatrix} 6 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-3} \\ M_{n-2} \\ M_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} y_1 - 2y_2 + y_3 \\ y_2 - 2y_3 + y_4 \\ y_3 - 2y_4 + y_5 \\ \vdots \\ y_{n-4} - 2y_{n-3} + y_{n-2} \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \end{bmatrix}$$

Hosil bo‘lgan matritsada $n-2$ ta noma’lum $(M_2, M_3, \dots, M_{n-2}, M_{n-1})$ hamda $n-2$ ta tenglik mavjud. Shu orqali, M_2 dan M_{n-1} gacha bo‘lgan noma’lum qiymatlar aniqlanadi.

Kubik splayn interpolyatsiyasining yuqoridagi shartlari bo‘yicha aniqlangan M_i qiymatlari asosida $S(x)$ funksiyasining noma’lum koeffitsiyentlari hisoblanadi. Natijada, bir nechta nuqtalar orqali o‘tadigan uchinchi darajali uzluksiz egri chiziq funksiyasi topiladi.