Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model

Abstract

This methodological framework examines the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model for estimating the conditional variance of heteroskedastic time series with autoregressive dynamics. In the GARCH framework, the model parameters are estimated using the maximum likelihood method. The objective of this article is to improve the assessment of volatility in the country's financial indicators through the application of appropriate econometric models.

Keywords: autoregressive, heteroskedastic, Kalman filter, maximum likelihood, residual, conditional variance.

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Generalized Autoregressive Conditional Heteroskedasticity (GARCH) Model

Autoregressive Conditional Heteroskedasticity (ARCH) model has been extended to the GARCH model[1]. The model determines the conditional variances of heteroskedastic[2] and autoregressive[3] indicators. Furthermore, the model is based on conditional variances, which are expressed as a linear function of the squared values of past time series.

According to the GARCH(*p, q*) model, the following is done:

(1) $E(\epsilon_t \mid \epsilon_u, u < t) = 0, \qquad t \in Z.$

(2) $\sigma_t^2 = Var(\epsilon_t \mid \epsilon_u, u < t) = \omega + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_i \sigma_{t-j}^2,$

(3) $t \in Z \; \epsilon_t = \sigma_t n_t$

Where, ϵ is the residual, σ^2 is a conditional variance, α and β are non-negative constants, ω is a strictly positive constant, p and q are lags representing the previous values of ARCH and GARCH, n is an independent and identically distributed random variable with mean zero and unit variance.

GARCH (1, 1) model is considered a special case of GARCH (p, q) and has the following form (p=q=1):

$$\begin{cases} \sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \\ \epsilon_t = \sigma_t n_t \end{cases}$$

Where, σ^2 is a conditional variance, α and β are non-negative constants, ω is a strictly positive constant, ϵ is the residual, n_t is an independent and identically distributed random variable with mean zero and unit variance.

GARCH parameters are estimated using the maximum likelihood function.

[1] Francq, C., & Zakoian, J.-M. (2010). GARCH models: structure, statistical inference, and financial applications.

[2] Heteroskedasticity is the condition where the variance of random variables is not constant across observations.

[3] The autoregressive indicator's residual of the current period correlates with its past values.

Maximum likelihood function

If the observations of the log return series $\{r_1, r_2 \dots, r_T\}$ are fixed, their joint probability density function with the parameter vector θ $(\omega, \alpha, \beta\ )$ can be represented as follows:

$$f(r_t, r_{t-1}, \dots, r_1; \theta) = f(r_t|\, r_{t-1}, \dots, r_1; \theta) \times \dots \times f(r_2|\, r_1; \theta) \times f(r_1; \theta)$$

The general likelihood function $L(\theta; r_t)$ can be derived based on the above equation:

$$L(\theta; r_t) = \prod_{t=1}^{T} f(r_t|\, r_{t-1}, \dots, r_1; \theta).$$

The log-likelihood function is more common to use in practice than the general likelihood function:

$$l(\theta; r_t) = \prod_{t=1}^{T} f(r_t|\, r_{t-1}, \dots, r_1; \theta) = \sum_{t=1}^{T} \log f(r_t|\, r_{t-1}, \dots, r_1; \theta).$$

Since the residual (ϵ_t) is important in estimating the GARCH parameters, it is appropriate to use the residual in the above equation. Under the distributional assumption of standard normal n_t in the conditional log-likelihood functions for GARCH(*p*, *q*) model with normal innovations, $\epsilon_t = \sigma_t n_t \sim N(0; \sigma_t^2)$ can be approached. Using the data up to time $t-1$ and based on $n_t \sim N(0; 1)$ the density function of ϵ_t is expressed as follows:

$$f(n_t|\, n_{t-1}, \dots, n_0; \theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{n_t^2}{2}\right).$$

Also, the density function of , ϵ_t is expressed as:

$$f(n_t|\, n_{t-1}, \dots, n_0; \theta) = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\sigma_t^2}\right).$$

The conditional log-likelihood function for the GARCH(*p, q*) model is calculated as follows:

$$l(\theta; \epsilon_{t-1}, \dots, \epsilon_0) = \sum_{t=1}^{T} \log f(\epsilon_t | \epsilon_{t-1}, \dots, \epsilon_0) = \sum_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{\epsilon_t^2}{2\sigma_t^2}\right) =$$

$$\sum_{t=1}^{T} \left[-\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\sigma_t^2) - \frac{\epsilon_t^2}{2\sigma_t^2}\right] =$$

$$-\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \left[\ln(\omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2) + \frac{\epsilon_t^2}{\omega+\alpha\epsilon_{t-1}^2+\beta\sigma_{t-1}^2}\right].$$

The values of the parameters that maximize the logarithmic probability are determined by taking special derivatives with respect to the unknown parameters (ω, α, β) and setting them equal to zero.